# ON THE NONSTEADY PROPAGATION OF CRACKS

**(O NEUSTANOVIVSHEMSIA RASPROSTRANENII TRESHCHIN)**

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Investigations of the processes of crack propagation have continued now for a considerable period of time, and it would be fair to say that in the field of stationary propagation of cracks the investigations have more or less reached completion.

One of the simplest problems of nonstationary propagation of cracks would appear to be the problem of the widening at constant velocity of a rectilinear crack in a uniform stress field perpendicular to the line of the crack. This problem has been investigated by a number of authors, starting with Mott [1], but it was not until the paper by Broberg [2] that it was treated as a problem of the dynamic theory of elasticity. Broberg, however, neglected the effect of cohesive forces, and for this reason came to the conclusion that the uniform propagation of cracks can take place only at a velocity equal to the velocity of propagation of Rayleigh surface waves: at any other velocity an uncompensated singularity occurs in the stress field at the end of the crack.

The present paper investigates on the basis of certain assumptions the effect of cohesive forces and derives an equation which defines the velocity of propagation of a crack in terms of the applied stress. It is shown that for every material there is a certain minimum velocity of stable uniform crack propagation. It is also shown that the velocity of stable propagation of a crack increases with increase in the splitting force and tends to the Rayleigh velocity: it would appear that in isotropic bodies the formation of a regime of uniform propagation at the Rayleigh velocity is prevented by the occurrence of branching of the crack.

**1. Fundamental concepts and hypotheses.** The problem is as follows. An infinite homogeneous and isotropic ideally brittle elastic body is subjected to a constant tensile stress (Fig. 1). At the initial instant of time a cut of length $2l_0$, which is greater than the critical, is made in the material, so that a crack immediately starts to grow. We shall assume that plane deformation takes place, and if we consider the motion when $t \gg l_0/c$, where $c$ is the velocity of propagation of transverse waves, then the initial non-uniformity, associated with the effect of the perturbation wave originating from the crack, need no longer be considered.

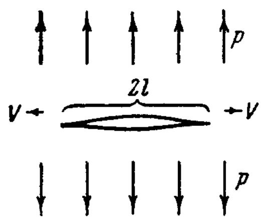


Fig. 1.

We can assume, therefore, that at some stage widening of the crack takes place* at a constant velocity $V$. In view of the comparative brevity of the period of non-uniform propagation we can also assume that uniform widening of the crack takes place from the start, so that the half-length $l$ of the crack is given by the expression

$$l = Vt \tag{1.1}$$

Following the procedure which has been adopted for static cracks [3, 4], we divide the surface of the crack $(-l \leqslant x \leqslant l)$ into two parts: an inner region $(-l + d \leqslant x \leqslant l - d)$ and an end region $(-l \leqslant x \leqslant -l + d,$ $l - d \leqslant x \leqslant l)$.

The surface of the inner region is considered to be free of stress, since the opposite sides of the crack in this region are a considerable distance apart; the end region, however, is subjected to cohesive forces distributed according to some particular law. In a theoretical investigation of quasi-brittle materials, in which the propagation of cracks is accompanied by plastic strains in a narrow layer adjacent to the surface of the crack, the surface of the crack must be taken as the boundary between the elastic and plastic regions, and the cohesive forces must include the forces exerted by the material in the plastic zone immediately in front of the leading edge of the crack on the material in the elastic state [4].

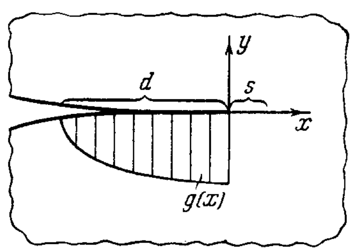


Fig. 2.

With respect to the end region and the distribution of cohesive forces

* It will be shown later that uniform widening of a crack at sub-Rayleigh velocities cannot proceed ad infinitum.

within it, we shall make two assumptions:

1. The end region of the surface of the crack widens at a constant velocity $v$ which is independent of the applied loading and much less than the velocity of widening of the crack. Thus

$$d = vt \qquad (v \ll V) \tag{1.2}$$

2. The distribution of cohesive forces $g(x)$ in the end region of the surface of the crack is autonomous, i.e. it is independent of the applied loading and depends only on the instantaneous length of the end region $d$, the distance from a point in the end region to a point on the crack $r = |l - x|$ and on the characteristics of the material: the elasticity modulus $E$, Poisson's ratio $\nu$ and the velocity of propagation of transverse waves $c$.

For quasi-brittle materials the yield point $\sigma_0$ is also of significance. From considerations of dimensional similarity we obtain

$$g(x) = EG\left(\frac{r}{d}, \nu\right) \tag{1.3}$$

where the function $G$ is universal. These assumptions define the autonomous nature of the development of the end region of the crack.

**2. The connection between the applied stress and the crack propagation velocity.** Let us consider a small region at the end of the crack (Fig. 2). We divide the stress field at every point in the elastic body into two parts: one determined solely by the cohesive forces and one evaluated ignoring these forces. In view of the slow rate of widening of the end region of the crack and the fact that $t \gg l_0/c$, the stress field set up solely by the cohesive forces near the ends of the crack can be considered to be quasi-stationary. It is shown in [5] that in the case of stationary propagation the distribution of the tensile stress $\sigma_y^{(1)}$ induced solely by cohesive forces is given by

$$\sigma_y^{(1)} = -\frac{1}{\pi\sqrt{s}}\int_0^d \frac{g\,dr}{\sqrt{r}} \tag{2.1}$$

over the remainder of the crack, where $s$ is the small distance to the end of the crack. Making use of (1.3), we obtain

$$\sigma_y^{(1)} = -\frac{E}{\pi\sqrt{s}}\int_0^d \frac{G(r/d, \nu)\,dr}{\sqrt{r}} = -\frac{E\sqrt{vt}}{\pi\sqrt{s}}\int_0^1 \frac{G(u, \nu)\,du}{\sqrt{u}} \tag{2.2}$$

Since the function $G(u, \nu)$ is universal, the integral on the right-hand side of (2.2) is independent of the applied loading.

Thus the quantity

$$E\sqrt{v}\int_0^1 \frac{G(u, \nu)\,du}{\sqrt{u}} \tag{2.3}$$

represents a constant characteristic of the material which we shall denote by $R$. The dimensions of the constant $R$ are

$$[R] = [F]L^{-3/2}T^{-1/2} = ML^{-1/2}T^{-5/2}$$

We can therefore reduce Expression (2.2) to the form

$$\sigma_y^{(1)} = -\frac{R}{\pi}\sqrt{\frac{t}{s}} \tag{2.4}$$

It has been shown by Broberg [2] that the distribution of tensile stresses $\sigma_y^{(2)}$ evaluated without taking into account cohesive forces has the following form over the remainder of the crack close to the end

$$\sigma_y^{(2)} = pF(m, \nu)\sqrt{\frac{ct}{s}}, \qquad m = \frac{V}{c} \tag{2.5}$$

where $p$ is the tensile stress at infinity, the nondimensional function $F(m, \nu)$ being defined by the expressions

$$F(m, \nu) = \frac{\sqrt{1-k^2m^2}\,\{4\sqrt{(1-k^2m^2)(1-m^2)} - (m^2-2)^2\}}{f(m, \nu)\,m^{3/2}\sqrt{2}} \tag{2.6}$$

$$f(m, \nu) = [(1-4k^2)m^2 + 4k^2]\,K(\sqrt{1-k^2m^2}) -$$
$$-4(1-k^2m^2)\,K(\sqrt{1-m^2}) - \frac{1}{m^2}[m^4 - 4(1+k^2)m^2 + 8]\,E(\sqrt{1-k^2m^2})$$
$$+\frac{8}{m^2}(1-k^2m^2)\,E(\sqrt{1-m^2}) \qquad \left(k = \sqrt{\frac{1-2\nu}{2-2\nu}}\right) \tag{2.7}$$

We now make the requirement that the stresses at the end of the running crack are finite. This means that the quantity $\sigma_y = \sigma_y^{(1)} + \sigma_y^{(2)}$ must be finite as $s \to 0$, whence, from (2.4) and (2.5), we obtain the basic relation

$$\frac{p\sqrt{c}}{R} = \frac{1}{\pi F(m, \nu)} \tag{2.8}$$

defining the propagation velocity in terms of the applied stress $p$. Expression (2.8) contains the constants of the material $c$, $\nu$, $R$. Note that for high velocities it seems reasonable to suppose that the distribution of cohesive forces $g(x)$ depends also on the velocity $V$ of the crack propagation, in which case the universal function $G$ assumes a new argument $m = V/c$ and the characteristic $R$ becomes dependent on $m$. However, for the range of velocities under consideration, which are less than the velocity at which branching of the crack commences, this dependence can as a first approximation be ignored.

**3. Discussion of the results.** Expression (2.8) is shown graphically in Fig. 3 for various values of $\nu$. It will be seen that for sufficiently small tensile stresses $p$ Equation (2.8) has no solution, so that the assumed regime of uniform propagation does not exist. An explanation of this phenomenon will be given later. For values of $p$ greater than the critical, corresponding to a minimum value of the right-hand side of (2.8), there will be two values of $m$ which satisfy Equation (2.8) for every value of $p$. One of them, the smaller, corresponds to an unstable propagation of the crack, since with increase in load the propagation velocity decreases; the other corresponds to stable propagation, since the propagation velocity increases with increase in load. It follows that the larger value of $m$ must be assumed, so that for a given material there exists a minimum velocity of uniform crack propagation.

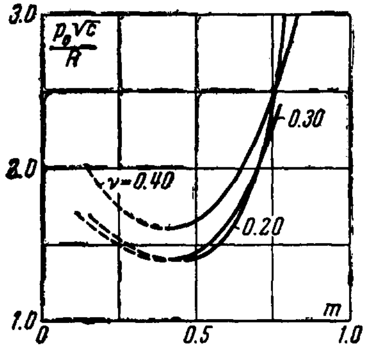


Fig. 3.

Note that the duration of the regime of uniform propagation under discussion is limited. In fact the "development" of the end region of the crack continues only until its resistance reaches a maximum corresponding to stationary propagation. It is well-known [5] that in the case of stationary propagation of a crack the tensile stress $\sigma_y$ at a distance $s$ from the end which is large compared with the size $d$ of the end region, but small compared with the characteristic dimension of the crack, can be expressed in the form

$$\sigma_y = \frac{N_0}{\sqrt{2s}}, \quad N_0 = \frac{\sqrt{2}K}{\pi} \tag{3.1}$$

where $K$ is the cohesion modulus [3,4]. In a process of nonsteady dynamic propagation the distribution of the stresses $\sigma_y$ on the remainder of the crack close to the end can, from (2.5) and (2.8), be written in the form $N_0/\sqrt{(2s)}$, but $N_0$ is now defined by the expression

$$N_0 = pF(m, \nu)\sqrt{2ct} = \frac{R}{\pi}\sqrt{2t} \tag{3.2}$$

Thus $N_0$ as defined by Formula (3.2) can increase only until it reaches a value $\sqrt{(2)}K/\pi$, i.e. for $t < T$, where

$$T = K^2 / R^2 \tag{3.3}$$

Uniform propagation is possible, therefore, only within the interval of time

$$l_0 / c \ll t < T \tag{3.4}$$

Now the function $F(m, \nu)$ vanishes when $m = 0$ and when $m = m_0$, where $m_0$ corresponds to the velocity of propagation of Rayleigh surface waves, so that the right-hand side of (2.8) becomes infinite for these values of $m$. Thus, by virtue of the instability of propagation of a crack with low velocity, as the loading increases, the propagation velocity must approach the Rayleigh velocity. It is well known, however, [6] (see also [5]), that in an isotropic body the Rayleigh velocity of rectilinear stationary propagation is never attained, since by this time branching of the crack has already taken place. A similar phenomenon evidently occurs also for nonstationary widening of a crack. We shall explain now why no regime of uniform propagation is produced for small values of $p$. In actual fact the half-length $l_0$ of the crack initially must be greater than the critical half-length $l_0^*$ corresponding to the given value of $p$ and equal to (see, for example, [4])

$$l_0^* = \frac{2K^2}{\pi^2 p^2} \tag{3.5}$$

Thus, as $p$ decreases the time $l_0/c$ increases and finally becomes comparable with the time of development of the end region $T$. The period of uniform propagation therefore decreases, and for a minimum value of $p$ equal to $p_m$ it becomes zero. Let us find the value of the ratio $cT/l_0$ for $p = p_m$ when the period of uniform propagation vanishes. Making use of (2.8), (3.3) and (3.5), we obtain

$$\frac{cT}{l_0} = \frac{cK^2\pi^2 p_m{}^2}{2R^2K^2} = \frac{c\pi^2 p_m{}^2}{2p_m{}^2\pi^2 F_m{}^2 c} = \frac{1}{2F_m{}^2} \tag{3.6}$$

where $F_m$ is the maximum value of the function $F(m, \nu)$ for a given $m$. We see from Fig. 3 that $F_m$ has the value $\sim 1/1.5\pi \sim 0.2$, so that $cT/l_0 \sim 10$. For $p > p_m$ we obtain evidently

$$10\frac{l_0}{c} < t < T \tag{3.7}$$

for the time of duration of uniform propagation.

For $t > T$ the cohesive forces are incapable of maintaining uniform propagation. The velocity of widening of the crack increases until it reaches the branching velocity, after which rectilinear propagation ceases. If the material is anisotropic the crack cannot start to branch, so that the velocity of widening will increase until a regime of uniform propagation at the Rayleigh velocity is attained.

**4. The experiments of Wells and Post.** Determination of the constant $R$. Wells and Post have published the results of some very interesting experiments on the nonsteady dynamic propagation of cracks, the conditions of which correspond reasonably closely to the theoretical presentation of this paper. The investigations of Wells and Post were carried out as follows (Fig. 4). A small initial cut was made along the perpendicular in the centre of one side of a rectangular plate made of transparent material type CR-39. The two edges of the plate parallel to the cut were fixed in rigid metal grips which moved apart in a direction perpendicular to the crack, so that the crack started to widen. The running crack was exposed to four successive flashes of polarized light and photographed; the isochromatics so obtained enabled the stress distribution close to the ends of the running crack to be deduced. Since the cohesive forces effect the stress distribution only at distances from the end of the crack of the order of several times the length of the end region, these experiments enable the coefficient of stress intensity $N_0$ to be found. This was done by Irwin [8,9]; the values of $N_0^2$ as a function of the length of crack $l$ are shown in Fig. 5 (denoted by the circles); the horizontal strokes indicate the mean velocity of propagation in the various sections. It will be seen that a reasonable approximation is to take the mean propagation velocity as constant, and the magnitude of $N_0^2$ as proportional to the length of the crack.

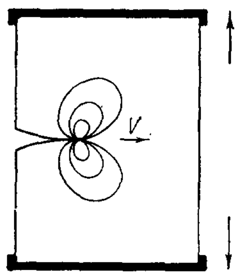


Fig. 4.

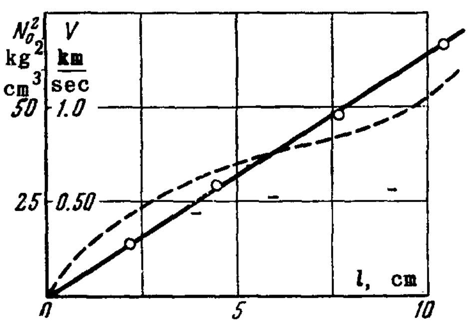


Fig. 5.

On the basis of (1.1) and (3.2) we have that

$$N_0^2 = \frac{R^2}{\pi^2}\frac{l}{V} \tag{4.1}$$

In Wells' and Post's experiments the magnitude of the time $l_0/c$ was of the order of $0.5 \times 10^{-5}$ sec ($l_0 \sim 0.3$ cm, $c \sim 0.8 \times 10^5$ cm/sec), and the time taken for the crack to propagate the full width of the plate was approximately $2.5 \times 10^{-4}$ sec. Let us evaluate the time $T$ required for the development of the end region. We have from (3.3) and

(4.1) that

$$T = \frac{K^2}{R^2} = \frac{K^2 l}{\pi^2 N_0^2 V} \tag{4.2}$$

For the present purpose we shall take the data corresponding to the second point to the right $N_0^2 = 0.45 \times 10^2$ kg$^2$/cm$^3$; $l = 7.8$ cm and a mean velocity $V = 0.5 \times 10^5$ cm/sec.

For materials of the type CR-39, as used in Wells' and Post's experiments, the cohesion modulus $K$ under static conditions is of the order of several hundred kg/cm$^{3/2}$. If we set $K = 100$ kg/cm $^{3/2}$ we obtain $T = 3 \times 10^{-3}$ sec, and the inequality (3.7) is satisfied. We can therefore assume that in spite of the smallness of the plate the proposed theory is applicable to conditions such as those in the experiments of Wells and Post. The data given in Fig. 5 enable the value of $R$ to be found for the material CR-39. If we draw a straight line through the origin of coordinates in Fig. 5 and as close as possible, from the point of view of the mean square deviation, to all four experimental points, we see from (4.1) that the slope of this line is $R^2/\pi^2 V$.

Taking the mean velocity as $V = 0.5 \times 10^5$ cm/sec, we find that $R = 1.6 \times 10^3$ kg/cm$^{3/2}$ sec$^{1/2}$. Irwin [8,9] attempted to make a quasi-static interpretation of Wells' and Post's experiments. The resulting curve is shown dashed in Fig. 5. This curve, however, was derived on the basis of the somewhat unrealistic supposition that the longitudinal dimension of the plate increases with time, the rate of increase being chosen to give the closest agreement with experiment. It appears that the interpretation put forward by Irwin is inadequate.

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